Multipoint Guidance—An Efficient Implementation of Predictive Guidance

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For intercept or rendezvous of a nonmaneuvering target, application of modern control theory leads naturally to the guidance policy called predictive guidance. The paper reviews the theory of predictive guidance and presents the results in a more complete form than was previously available. A new method of implementing predictive guidance, called multipoint guidance, is presented. In the context of command guidance for intercepting a ballistic re-entry vehicle, multipoint guidance has been found to yield miss-distances similar to those of a standard implementation of predictive guidance while requiring much less real-time data processing.

Predictive Guidance

THE problem of intercept or rendezvous of a nonmaneuvering target may be formulated as follows. The nonlinear dynamic equations for the intercept (I) and target (T) vehicles are, respectively,

$$\dot{y}_I = f_I(y_I, u, t), \quad y_I(t_0) \text{ known} \qquad \dot{y}_T = f_T(y_T, t), \quad y_T(t_0) \text{ known} \tag{1}$$

where y_I and y_T are the vehicle state vectors, t represents time, t_0 is the initial time, and $u \varepsilon U$ is the control vector of I. The control is to be chosen to optimize the performance index

$$\min_{u(t)\varepsilon U, t_f} J \tag{2}$$

where the problem is either an intercept or a rendezvous according to the choice of J.

For most applications it is necessary in addition to model process noise terms which result in uncertainty about the future vehicle states as well as measurement noise terms which result in uncertainty about the present states. These complexities are not included here since a separation of estimation and control is assumed. The focus herein is on the deterministic control problem.

Theory of Predictive Guidance

Predictive guidance[1,2] is derived from a modification of the above nonlinear problem [Eqs. (1) and (2)]. The modification amounts to a particular form of linearization. It is shown in the next section how the solution to the modified problem is applied to obtain an approximate solution of the actual nonlinear problem. The modified dynamic equations used to derive predictive guidance are

$$\begin{aligned} \dot{x}_I &= v_I \\ \dot{v}_I &= T(t) - D(t) + a_I(t) \\ \dot{x}_T &= v_T \\ \dot{v}_T &= a_T(t) \end{aligned} \tag{3}$$

where x_I, x_T are the 3-dimensional Cartesian position vectors for I and T, v_I, v_T are the 3-dimensional Cartesian velocity vectors for I and T, $T(t)$ is the thrust applied to I, $D(t)$ is the drag applied to I, and $a_I(t)$, $a_T(t)$ are the 3-dimensional Cartesian acceleration vectors for I and T. It is assumed that $T(t)$, $D(t)$, and $a_T(t)$ are known functions of time. The acceleration vector $a_I(t)$ is assumed to be the control for I. The Eqs. (3) can be written in the form

$$\begin{pmatrix} \dot{v} \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ I_3 & 0 \end{pmatrix} \begin{pmatrix} v \\ x \end{pmatrix} + \begin{pmatrix} a \\ 0 \end{pmatrix} - \begin{pmatrix} a_I \\ 0 \end{pmatrix} \tag{4}$$

where

$$\begin{aligned} v &= v_T - v_I \\ x &= x_T - x_I \\ a &= a_T(t) - [T(t) - D(t)] \end{aligned} \tag{5}$$

I_3 is a 3×3 unit matrix. A quadratic performance index is defined as follows:

$$J = \tfrac{1}{2}x'(t_f)M_1 x(t_f) + \tfrac{1}{2}v'(t_f)M_2 v(t_f) + \frac{1}{2}\int_{t_0}^{t_f} a_I'(t) R a_I(t)\, dt \tag{6}$$

For simplicity it is assumed that M_1, M_2, and R, are of the form

$$M_1 = c_1 I_3, \quad M_2 = c_2 I_3, \quad R = I_3 \tag{7}$$

where c_1 and c_2 are scalars. Equations (4) and (6) are in the form of a standard linear quadratic optimal control problem for which the solution is[1]

$$a_I = S_1 v + S_2 x + k_a$$

where

$$a_I, v, x, k_a \text{ are 3-vectors} \qquad S_1, S_2 \text{ are symmetric } 3\times 3 \text{ matrices} \tag{8}$$

The equations for S_1, S_2, and k_a are

$$\begin{bmatrix} \dot{S}_1 & \dot{S}_2 \\ \dot{S}_2' & \dot{S}_4 \end{bmatrix} + \begin{bmatrix} S_2 & 0 \\ S_4 & 0 \end{bmatrix} + \begin{bmatrix} S_2' & S_4 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} S_1 S_1 & S_1 S_2 \\ S_2' S_1 & S_2' S_2 \end{bmatrix} = 0 \tag{9}$$

$$S_1(t_f) = c_2 I_3, \quad S_2(t_f) = 0, \quad S_4(t_f) = c_1 I_3 \tag{10}$$

$$\begin{bmatrix} \dot{k}_a \\ \dot{k}_b \end{bmatrix} + \begin{bmatrix} S_1 a \\ S_2' a \end{bmatrix} + \begin{bmatrix} k_b \\ 0 \end{bmatrix} - \begin{bmatrix} S_1 k_a \\ S_2' k_a \end{bmatrix} = 0 \tag{11}$$

$$k_a(t_f) = 0, \qquad k_b(t_f) = 0 \tag{12}$$

where S_4 is a 3×3 matrix and k_b is a 3-vector.

Equations (9) and (10) are solved by

$$S_1 = s_a I_3, \quad S_2 = s_b I_3, \quad S_4 = s_d I_3 \tag{13}$$

provided that the scalars s_a, s_b, and s_d satisfy

$$\begin{aligned} \dot{s}_a + 2s_b - s_a^2 &= 0, \quad s_a(t_f) = c_2 \\ \dot{s}_b + s_d - s_a s_b &= 0, \quad s_b(t_f) = 0 \\ \dot{s}_d - s_b^2 &= 0, \quad s_d(t_f) = c_1 \end{aligned} \tag{14}$$

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After eliminating s_d, the solution for s_a and s_b is

$$s_a = A/D, \qquad s_b = B/D \tag{15}$$

where

$$\begin{aligned} A &= 1/c_1 + (1/c_2)(t_f - t)^2 + \tfrac{1}{3}(t_f - t)^3 \\ B &= 1/c_2(t_f - t) + \tfrac{1}{2}(t_f - t)^2 \\ D &= 1/c_1 c_2 + (1/c_1)(t_f - t) + (1/3c_2)(t_f - t)^3 + \tfrac{1}{12}(t_f - t)^4 \end{aligned} \tag{16}$$

Using Eqs. (11, 15, and 16) the following equations can be derived for the elements of k_a:

$$\begin{gathered} \ddot{k}_{a_i} - s_a \dot{k}_{a_i} + (s_b - \dot{s}_a) k_{a_i} + (\dot{s}_a a_i + s_a \dot{a}_i - s_b a_i) = 0 \\ k_{a_i}(t_f) = 0, \quad \dot{k}_{a_i}(t_f) = -c_2 a_i, \quad \text{for} \quad i = 1, 2, 3 \end{gathered} \tag{17}$$

The solution for k_{a_i} is

$$k_{a_i} = [s_a - (t_f - t) s_b] L_{1_i} + s_b L_{2_i}$$

where

$$\begin{aligned} L_{1_i} &= \int_t^{t_f} a_i(\tau)\, d\tau \\ L_{2_i} &= \int_t^{t_f} \int_t^{\sigma} a_i(\tau)\, d\tau \cdot d\sigma, \quad \text{for} \quad i = 1, 2, 3 \end{aligned} \tag{18}$$

To summarize the results, from Eqs. (8, 13, and 18)

$$a_I(t) = s_b x(t_f, t) + [s_a - s_b(t_f - t)] v(t_f, t)$$

where

$$\begin{aligned} x(t_f, t) &= x(t) + v(t)(t_f - t) + \int_t^{t_f} \int_t^{\sigma} a(\tau)\, d\tau\, d\sigma \\ v(t_f, t) &= v(t) + \int_t^{t_f} a(\tau)\, d\tau \end{aligned} \tag{19}$$

The vectors $x(t_f, t)$ and $v(t_f, t)$ are called, respectively, the "predicted miss-distance" and the "predicted miss-velocity" of T and I at time t_f. They are the miss-distance and miss-velocity expected at the terminal time t_f, based on the state at the present time t and under the assumption of zero future interceptor control acceleration (i.e., $a_I(\tau) = 0$, $t \leqq \tau \leqq t_f$). The result summarized by Eq. (19) is intuitively appealing. Namely, the optimal control at time t is to accelerate the intercept vehicle in a direction which is a weighted average of the predicted miss-distance and miss-velocity at time t_f. The constants of proportionality are given by

$$s_b = \frac{(1/c_2)(t_f - t) + \frac{1}{2}(t_f - t)^2}{(1/c_1 c_2) + (1/c_1)(t_f - t) + (1/3c_2)(t_f - t)^3 + \frac{1}{12}(t_f - t)^4} \tag{20}$$

$$s_a - s_b(t_f - t) = \frac{(1/c_1) - \frac{1}{6}(t_f - t)^3}{(1/c_1 c_2) + (1/c_1)(t_f - t) + (1/3c_2)(t_f - t)^3 + \frac{1}{12}(t_f - t)^4} \tag{21}$$

Recall that the performance index used in deriving these controls is, from Eqs. (6) and (7)

$$J = \tfrac{1}{2} c_1 x'(t_f) x(t_f) + \tfrac{1}{2} c_2 v'(t_f) v(t_f) + \tfrac{1}{2} \int_{t_0}^{t_f} a_I'(t) a_I(t)\, dt \tag{22}$$

For rendezvous, with a non-negligible weight on the integral control cost, the correct time-varying constants of proportionality are given by Eqs. (20) and (21). If no weight is desired on the control cost, c_1 and c_2 may be allowed to increase without limit, yielding

$$a_I(t) = \frac{6}{(t_f - t)^2} x(t_f, t) - \frac{2}{(t_f - t)} v(t_f, t) \tag{23}$$

For intercept problems with no cost on terminal miss-velocity, the constant c_2 is set equal to zero, yielding

$$a_I(t) = \frac{c_1(t_f - t)}{1 + (c_1/3)(t_f - t)^3} x(t_f, t) \tag{24}$$

If in this case no weight is desired on the integral cost term then c_1 may be allowed to increase without limit, yielding

$$a_I(t) = \frac{3}{(t_f - t)^2} x(t_f, t) \tag{25}$$

When no cost is employed on interceptor control as in Eqs. (23) and (25) the gain terms become very large as the time-to-go becomes small. This is natural since a large acceleration is needed to have a significant effect on the interceptor terminal position and velocity when the time-to-go is small. In a practical guidance problem, when the vehicle states are stochastic, the achieved control acceleration normally saturates at the acceleration limit of the interceptor shortly before the terminal time.

Note that Eq. (25) can be interpreted as proportional navigation for the case of very simple vehicle dynamics where the target acceleration $a_T(t)$, the interceptor thrust $T(t)$, and the interceptor drag $D(t)$, are all zero.[3] In this event Eqs. (5) and (19) yield

$$x(t_f, t) = x(t) + v(t)(t_f - t) \tag{26}$$

and Eq. (25) becomes

$$a_I(t) = 3V\dot{\beta} \tag{27}$$

where V = closing velocity along the line-of-sight direction and $\beta \approx x(t)/V(t_f - t)$ is interpreted as the line-of-sight angle. This helps to explain why proportional navigation guidance laws typically perform well for intercepts when the vehicles have small accelerations other than the interceptor control acceleration (e.g., in space applications) and poorly for intercepts when the vehicles are subject to significant uncontrolled accelerations (e.g., typical atmospheric applications).

Standard Implementation of Predictive Guidance

In practice the interceptor may not be capable of accelerating in any desired direction as given by $a_I(t)$. In that case it is necessary in real time to choose the particular terminal time, $\hat{t}_f$, which leads to an achievable control acceleration, $a_I(t)$. The standard method of implementation is outlined below. It is assumed that a real time estimator provides the guidance law with estimates $\hat{x}_T$ and $\hat{x}_I$ of the actual vehicle states.

Computations for a guidance update (interceptor with lateral acceleration only)

1) The target position trajectory $\hat{x}_T(t \mid \sigma)$ at future times t is predicted on the basis of information to the present time, σ. Normally this is achieved by integrating the nonlinear target dynamic equations forward, starting at the latest state estimate. 2) The interceptor position trajectory $\hat{x}_I(t \mid \sigma)$ at future times t is predicted on the basis of information to the present time σ, and using zero interceptor control. Normally the trajectory is predicted by integrating the nonlinear interceptor equations forward

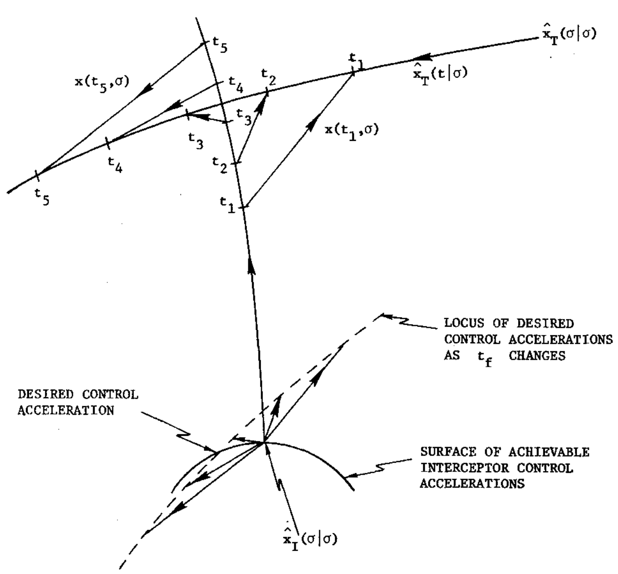

Fig. 1 Implementation of predictive guidance (lateral control only).

starting at the latest state estimate. 3) A search is made over terminal times t_f to find the time $\hat{t}_f$ which yields an achievable control acceleration. Figure 1 illustrates this process in two dimensions. If there is no time $\hat{t}_f$ yielding an achievable acceleration then an accurate intercept is not possible based on the latest state estimates. The time $\hat{t}_f$ can then be chosen so that the difference between the desired and achievable accelerations is minimized. 4) The optimal control, $\hat{a}_I(t)$, is obtained from the appropriate predictive guidance formula using $\hat{t}_f$ and $x(\hat{t}_f, \sigma)$ in place of t_f and $x(t_f, t)$. This control is applied until the guidance is again updated.

An initial nominal interceptor trajectory is needed to begin the above process. Normally this is provided by the prelaunch calculations which determine both the time of launch of the intercept vehicle and the launch angles. When the interceptor has both lateral and axial control it is possible to achieve intercepts at a prespecified terminal time or in a prespecified intercept plane. The above technique for implementing predictive guidance can be modified in a straightforward manner to handle these situations.

Comments on Predictive Guidance

It can now be seen why predictive guidance provides a near-optimal solution to the original nonlinear guidance problem [Eqs. (1) and (2)]. In deriving predictive guidance it was assumed that the nonlinear thrust and drag accelerations [$a_T(t)$, $T(t)$, and $D(t)$] were known time functions, a strategem which allowed an analytic solution for the optimal control. As might be expected, the optimal control [Eq. (19)] implicitly involves these functions. Implementation of predictive guidance thus requires real time determination of the actual functions through integrating the actual nonlinear vehicle state equations forward in time. The principal approximation inherent in this procedure for solving the original nonlinear guidance problem is the assumption that the interceptor acceleration due to drag, $D(t)$, does not change as the interceptor control changes. (No approximation is introduced by assuming that the target acceleration $a_T(t)$ and the interceptor thrust $T(t)$ remain unchanged with interceptor control changes.) The excellent experimental results commonly achieved with predictive guidance are apparently due to $D(t)$ being only moderately sensitive to control changes and to the feedback nature of the guidance process which automatically corrects errors as time proceeds. In fact it has been found that predictive guidance is relatively insensitive both to variations in the gain term which gives the magnitude of the control acceleration and to moderate approximations in the equations used to predict the target and interceptor future positions.

While predictive guidance defines an effective guidance policy, the standard method of implementation described above requires a considerable amount of real time computation to predict repeatedly the future positions of the target and interceptor even if only approximate prediction equations are used. This fact has motivated development of multipoint guidance.

Multipoint Guidance

Multipoint Controller Design

The technique that has been used in synthesizing multipoint guidance is called multipoint controller design.[5] This method has four sequential steps. 1) A number of representative trajectories spanning the state space of concern are generated. The dynamic states of the vehicles along these trajectories as well as the desired controls (specified by the guidance policy) are computed and stored. 2) A guidance law structure containing unspecified functions of the dynamic states is postulated. 3) Data from the representative trajectories are employed to determine the unspecified functions. (A least-squares fit is usually convenient if there are more representative trajectories than unspecified functions.) 4) The controller performance is tested over the state space of concern. Of particular interest is the accuracy with which the controller approximates the desired policy for dynamic states that are not on one of the representative trajectories. Assuming that the controller performance is not satisfactory it is necessary to return to step 2 and modify the guidance law structure. Success of the method depends critically on the designer's understanding of the control problem as reflected by his ability to specify a simple controller structure capable of approximating the desired policy. Note that the representative trajectories are not referred to in real time—they are used only in the design phase to determine the functions of states (which are referred to in real time).

Multipoint controller design is useful for direct real time controller design in applications for which a good control policy is known, but direct implementation leads to excessive computation requirements. In the remainder of the paper it is shown how the technique has been applied to designing a command guidance law for intercept of a ballistic re-entry vehicle by an aerodynamically controlled missile. This is an application for which the control policy provided by predictive guidance is known to yield good miss-distances but for which the standard implementation requires a considerable amount of computation.

Design of Multipoint Guidance

Desired direction of travel

A key element of the structure of the multipoint guidance law is the concept of the desired direction of travel, v_{ID}. This is a three-dimensional unit vector which points in the direction such that if the interceptor velocity v_I was also in that direction and the future interceptor control acceleration was zero the predicted miss-distance $x(t_f, t)$, would be minimized. Normally, but not always, the minimum predicted miss-distance is zero if there is freedom to alter instantaneously the interceptor direction of travel. For interceptor velocities, v_I, not in the same direction as v_{ID} the multipoint guidance law applies a control proportional to the angle between those vectors and in the direction tending to align v_I with v_{ID}. The proportionality constant is chosen so that the control acceleration approximates the predictive guidance policy. Figure 2 illustrates the geometry at time t in the plane defined by $v_I(t)$ and $v_{ID}(t)$. The angular error, $\Delta\phi$, is the angle between $v_I(t)$ and $v_{ID}(t)$. If the zero-control interceptor trajectory is extrapolated along the desired direction $v_{ID}(t)$ a zero miss-distance occurs with the predicted target position $\hat{x}_T(t_f \mid t)$ at some time t_f (from the definition of $v_{ID}(t)$ and assuming the minimum predicted miss-distance is zero). If on the other hand the zero-control interceptor trajectory is extrapolated along the actual interceptor velocity vector $v_I(t)$, the interceptor reaches the point $x_I(t_f)$ at the same time t_f. The predicted miss-distance at time t_f is $x(t_f, t)$, as shown in Fig. 2. From Eq. (24) the desired control acceleration is in the same direction as $x(t_f, t)$ with magnitude

$$|a_I(t)| = \frac{c_1(t_f - t)}{1 + (c_1/3)(t_f - t)^3}\,|x(t_f, t)| \tag{28}$$

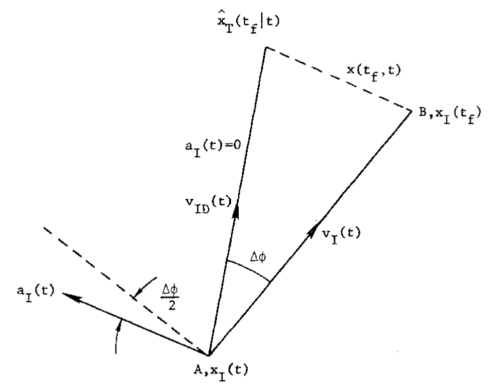

Fig. 2 Geometry of multipoint guidance.

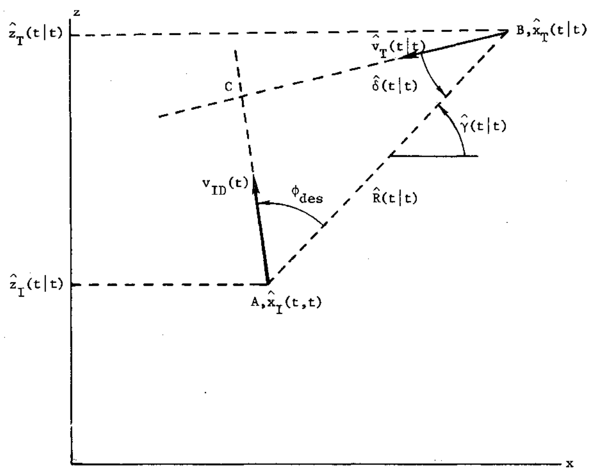

Fig. 3 Relative state variables of the target and interceptor.

Assuming that the zero-control trajectories are nearly straight lines of equal length

$$|x(t_f, t)| = \{2|AB|^2 - 2|AB|^2 \cos \Delta\phi\}^{1/2} \tag{29}$$

For small $\Delta\phi$ the following approximation can be made:

$$|x(t_f, t)| \approx |AB| \, \Delta\phi \tag{30}$$

Also the distance $|AB|$ can be approximated by

$$|AB| \approx |v_I(t)|(t_f - t) \tag{31}$$

assuming the average interceptor speed over the period $(t_f - t)$ is not appreciably different from the initial speed $|v_I(t)|$. Then from Eqs. (24, 30, and 31) the magnitude of the control acceleration is

$$|a_I(t)| = \frac{c_1(t_f - t)^2}{1 + (c_1/3)(t_f - t)^3} |v_I(t)| \, \Delta\phi \tag{32}$$

The multipoint guidance law applies a lateral acceleration [i.e., a commanded acceleration perpendicular to $v_I(t)$] which is in the plane of $v_I(t)$, and $v_{ID}(t)$, and with magnitude given by Eq. (32). Note that this commanded acceleration is not necessarily an achievable acceleration since a search has not been made over possible terminal times to find the terminal time $\hat{t}_f$ which yields an achievable acceleration (see step 3 of Computations for a Guidance Update). In other words, if the interceptor is commanded to perform the above lateral acceleration, an unavoidable additional acceleration results due to induced drag. This induced drag acceleration has not been accounted for in the equations. For interception of a ballistic re-entry vehicle this approximation and those of Eqs. (30) and (31) have been found to cause negligible deterioration of the miss-distance. This is apparently due to: 1) When the angle error $\Delta\phi$ is small the errors caused both by the approximation of Eqs. (30) and (31) and by neglecting induced drag become infinitesimal. 2) When the angle error $\Delta\phi$ is large the orientation of the control acceleration $a_I(t)$ remains consistent with the predictive guidance policy. Errors in the magnitude of the acceleration are unimportant because the acceleration limits of the interceptor dominate in this case. Thus the multipoint guidance law attempts to null out the angle error $\Delta\phi$ between the actual direction of interceptor travel and the direction that would result in zero predicted miss-distance. When the angle error is small the generated guidance commands are very close to those of predictive guidance. When the angle error is large the direction of the generated control acceleration is very close to that of predictive guidance. The magnitude of the generated control acceleration may differ considerably from that of predictive guidance but the actual magnitude of acceleration achieved by the vehicle is determined by the vehicle acceleration limits in this case.

While the miss-distance error is relatively insensitive to the gain term used in Eq. (32) it is of course sensitive to the accuracy of determining the direction of acceleration, i.e., to the direction of v_{ID}. A consistent error in this direction results in a curved interceptor trajectory and large miss-distances if the resulting acceleration requirement is beyond the capability of the interceptor. Accordingly it is desirable for each application to perform a sensitivity study to determine acceptable errors in the direction of v_{ID}. As an example, it was found in the context of intercepting a ballistic re-entry vehicle traveling at 14,000 ft/sec with an interceptor traveling at 7,000 ft/sec that a bias in the direction of v_{ID} up to 0.01 rad resulted in a minor total curvature of the interceptor trajectory ($\approx 2.5°$).[4] The acceleration requirement resulting from this curvature was well within the capability of the interceptor. Biases significantly greater than 0.01 rad were found to result in a significant requirement for control acceleration, thus reducing the interceptor capability to correct for other errors.

Determination of the desired direction of travel

The desired direction of travel is most easily specified in terms of the relative state variables. Figure 3 illustrates these variables in the plane defined by the present estimated target velocity vector, $\hat{v}_T(t|t)$, and the line joining the present estimated positions of the target and interceptor $\hat{x}_T(t|t)$, $\hat{x}_I(t|t)$. For the present application it has been found that the desired direction of travel may be taken to lie also in this plane. For a target re-entry vehicle with a low ballistic coefficient the actual intercept may not occur in this plane because the force of gravity causes the trajectory to curve downward. However, even in this case the true desired direction of travel is very nearly in the plane of $\hat{v}_T(t|t)$ and the line joining $\hat{x}_T(t|t)$, $\hat{x}_I(t|t)$, because gravity acts on both vehicles for an equal time, causing both trajectories to move downward to an almost equal extent.

If the predicted target and interceptor velocities were constant with magnitudes $|\hat{v}_T|$ and $|\hat{v}_I|$ the angle ϕ_{des} leading to zero predicted miss-distance would be

$$\phi_{des} = \arcsin \frac{|\hat{v}_T| \sin \hat{\delta}}{|\hat{v}_I|}, \quad \text{for } |\hat{v}_T| \sin \hat{\delta} \leq |\hat{v}_I| \tag{33}$$

from solution of the triangle ABC in Fig. 3. Note that there are two feasible solutions for ϕ_{des}, one in the range $0 \leq \phi_{des} \leq \pi/2$, and the other in the range $\pi/2 \leq \phi_{des} \leq \pi$.† For the case of $|\hat{v}_T| \sin \hat{\delta} > |\hat{v}_I|$ there is no solution for ϕ_{des} which leads to zero predicted miss-distance but there is a solution for ϕ_{des} which minimizes the predicted miss-distance.[4]

The predicted velocities of the vehicles may vary considerably over the interval $(t_f - t)$ but if estimates of the average velocities are available the value of ϕ_{des} is given by using these averages in Eq. (33). Simple approximations can be derived for the average velocities on the assumption that the present drag accelerations of the vehicles remain constant. Allowance can also be made for the thrust of the interceptor. The results are

$$|v|^{ave} = |v_0| - \frac{D_0 \bar{t}}{2m_0}, \quad \text{for no thrust}$$

$$|v|^{ave} = |v_0| + \frac{T - D_0}{m_1{}^2 \bar{t}} \left[m_1 \bar{t} + (m_0 - m_1 \bar{t}) \ln \left(\frac{m_0 - m_1 \bar{t}}{m_0} \right) \right] \tag{34}$$

for thrust over the entire interval

$$|v|^{ave} = |v_0| - \frac{D_0}{2m_2 \bar{t}} (\bar{t} - t_g)^2 + \frac{T - D_0}{m_1 \bar{t}} \left[t_g + \ln \left(\frac{m_0}{m_2} \right) \left(t - \frac{m_0}{m_1} \right) \right]$$

for thrust over an interval t_g, less than the entire interval, where $\bar{t} = t_f - t =$ time-to-go, $t_g =$ time-to-go for thrusting, $|v_0| =$ initial velocity, $D_0 =$ initial drag force, $T =$ thrust force, $m_0 =$ initial mass, $m = m_0 - m_1 t =$ mass when thrusting, $m_1 =$

† Normally the guidance law designer will choose the first solution corresponding to $0 \leq \phi_{des} \leq \pi/2$ since this leads to an earlier intercept time. However, there may be special circumstances where the second solution is preferred.

Table 1 Representative trajectories used in designing the multipoint guidance law[a]

	Re-entry vehicle characteristics				
Trajectory number	Re-entry angle at 400 kft altitude	Ballistic coefficient	Impact position relative to interceptor launch point	Interceptor launch elevation angle	Intercept altitude
1	40°	400 lb/ft²	+0 kft	41.5°	35 kft
2	40	400	−50	111	37
3	40	400	40	41.5	25
4	40	667	0	41.5	36
5	40	667	−50	110	33
6	40	667	0	41.5	26
7	40	2000	0	41.5	30
8	40	2000	−50	110	35
9	40	2000	0	41.5	24
10	25	400	0	30	30
11	25	400	−50	90	30
12	25	400	0	30	17
13	25	667	0	28	32
14	25	667	−50	90	20
15	25	667	0	28	16.5
16	25	2000	0	28	32
17	25	2000	−50	90	26
18	25	2000	0	28	16
19	10	400	0	25.2	21
20	10	400	−50	90	24
21	10	667	−50	54	22
22	10	2000	−50	90	14
23	25	667	30	20	10
24	25	2000	30	20	10
25	25	667	−40	54	22
26	25	2000	−40	54	20

[a] All trajectories are in a vertical plane. The target ballistic coefficient is constant.

rate of change of mass when thrusting, and m_2 = mass at completion of thrusting. An approximation for the time-to-go, suitable for use in Eqs. (34), is

$$\bar{t} \approx \frac{\hat{R}}{|\hat{v}_T|\cos\hat{\delta} + |\hat{v}_I|\left[1 - \left(\frac{|\hat{v}_T|\sin\hat{\delta}}{|\hat{v}_I|}\right)^2\right]^{1/2}} \tag{35}$$

To summarize the above, a candidate method for specifying ϕ_{des} (for the case when zero predicted miss-distance is achievable) is as follows. Equation (35) is employed to calculate $\bar{t}$. The appropriate equations from Eqs. (34) are then used to calculate $|v_T|^{\text{ave}}$, and $|v_I|^{\text{ave}}$, and finally ϕ_{des} is obtained from Eq. (33). In this manner ϕ_{des} is specified directly in terms of the present vehicle states. A guidance law based on these equations has been coded and tested in a one-on-one intercept simulation. It was found to yield acceptable miss-distances. However, improved accuracy and a slightly less complex structure were obtained by replacing Eq. (33) with a multipoint controller of form

$$\phi_{\text{des}} = c_0(\hat{R}) + c_1(\hat{R})a|a| \tag{36}$$

where

$$a = \frac{|\hat{v}_T|^{\text{ave}}\sin\hat{\delta}}{|\hat{v}_I|^{\text{ave}}} \tag{37}$$

$\hat{R}$ is the relative range (see Fig. 3).

In order to specify the arbitrary coefficients c_0 and c_1 a number of representative trajectories were generated through the state space. Table 1 lists the parameter variations chosen. Each representative trajectory consists of a deterministic target trajectory, and a deterministic zero-control interceptor trajectory which intercepts the target trajectory with zero miss-distance. The trajectories were all in a vertical plane. A constant ballistic coefficient was used in integrating the target trajectories. This is equivalent to assuming a constant ballistic coefficient equal to the latest estimate when predicting target trajectories in a standard implementation of predictive guidance.

Along each representative trajectory the relative states of the vehicles were computed and stored, as was the true desired direction of travel. Since there are more representative trajectories than arbitrary coefficients, a least-squares fit was made to the data at each value of R to obtain the best compromising values of c_0 and c_1. Figure 4 shows the values of $c_0(R)$ and $c_1(R)$ that were obtained for a hardsite interceptor. As the figure suggests, values of c_0 and c_1 were calculated at discrete values of R, with a linear interpolation being made at intermediate R values. It is clear from the fairly smooth shape of c_0 and c_1 that only a small amount of computer storage is needed to store these functions.

Table 2 shows the errors between the true values of ϕ_{des} and the values generated by Eqs. (34–37). The least-squares fit to obtain c_0 and c_1 was performed using only the representative trajectories marked with an asterisk in Table 2. The remaining trajectories were used as test trajectories. Table 2 also shows for comparison the errors obtained when Eq. (33) is substituted for Eq. (36). With both guidance law structures the accuracy generally improves as R becomes smaller. Fairly large errors (15–40 mrad) occur at long ranges with both guidance law structures on trajectories 2, 5, 8, 11, 14, 17, 20, and 22. Of these, trajectories 2, 5, 11, 14, and 20 are all low target ballistic coefficient trajectories. Because these trajectories are fairly slow and there is sufficient time to make corrections as the range reduces, errors of this magnitude have been found

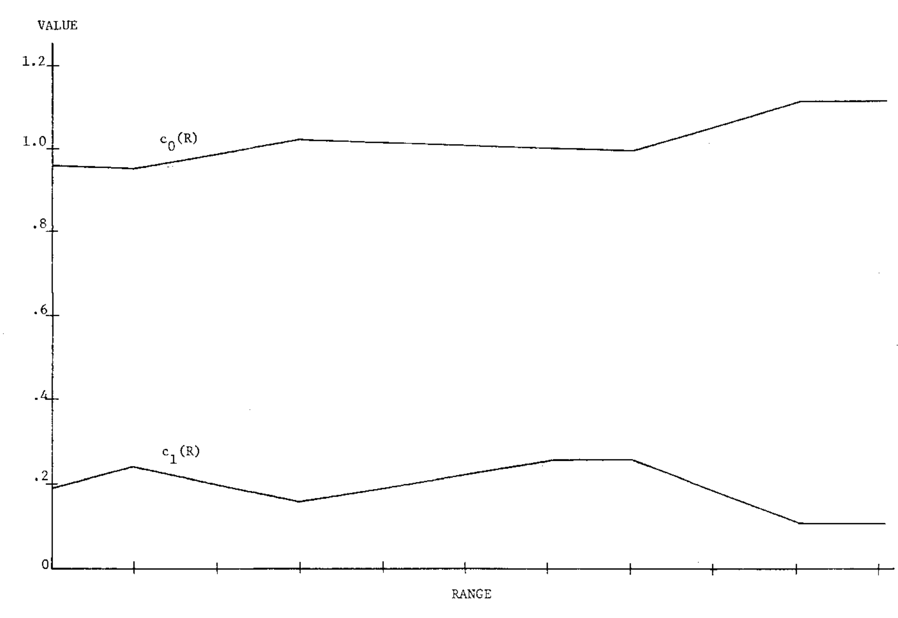

Fig. 4 Coefficients of the multipoint guidance law for a hardsite interceptor.

Table 2 Errors in obtaining the desired direction of travel (angle errors in milliradians)[a]

Number of Representative trajectory[b]	5000 ft ϕ_1	5000 ft ϕ_2	10,000 ft ϕ_1	10,000 ft ϕ_2	20,000 ft ϕ_1	20,000 ft ϕ_2	40,000 ft ϕ_1	40,000 ft ϕ_2	80,000 ft ϕ_1	80,000 ft ϕ_2
1	−0.3	−0.03	−0.3	−0.04	−0.03	−0.01	0.3	0.2	0.6	0.5
2	−4.9	4.9	−0.6	9.7	−23.8	17.6	−18.3	22.6	0.1	22.0
3	−0.2	−0.04	−0.3	−0.07	−0.07	−0.06	0.2	0.2	2.0	2.1
4*			−0.4	−0.04	−0.04	−0.03	0.2	0.07	0.3	0.2
5*	−0.3	5.1	4.7	9.7	−23.1	16.8	−10.3	28.1	21.4	40.5
6*	−0.2	−0.03	−0.2	−0.05	−0.04	−0.03	0.2	0.09	0.6	0.7
7*			−0.3	−0.04	−0.07	−0.06	0.03	−0.06	−0.02	−0.04
8*			15.8	8.2	−16.9	15.4	−0.3	30.8	59.3	79.8
9			−0.2	−0.04	−0.06	−0.05	0.01	−0.04	0.3	0.3
10	−0.4	−0.06	0.5	0.1	0.4	0.4	0.4	0.9	0.3	0.2
11	−7.2	4.5	−1.2	9.7	−10.4	24.0	45.1	83.9		
12*	−0.1	0.04	0.3	0.12	0.4	0.4	2.4	2.6		
13*	−0.6	−0.07	−0.7	−0.1	−0.2	−0.2	0.2	−0.09	0.4	0.2
14*	−7.7	4.9	−2.7	9.2	−17.8	22.0	19.2	62.0		
15*	−0.3	−0.06	−0.3	−0.1	−0.1	−0.1	0.3	0.3		
16*			−0.6	−0.1	−0.2	−0.2	0.04	−0.2	0.007	−0.3
17			−0.9	7.7	−22.2	19.3	4.1	46.1		
18			−0.1	−0.04	−0.04	−0.04	0.01	0.04	0.8	1.1
19	−0.02	−0.01	0.1	−0.03	0.3	0.3	1.0	1.3		
20	3.2	2.7	9.3	8.3	24.1	32.3				
21*	2.6	2.4	5.7	4.7	5.3	13.2				
22*	5.2	10.6	20.7	25.5	38.7	79.1				
23	−2.5	−0.10	−3.8	−2.0	−3.2	−3.8	−5.5	−7.8		
24*	−2.6	−0.9	−3.9	−1.7	−1.7	−3.2	−1.7	−5.4	1.1	−5.4
25*	1.1	2.0	3.8	3.7	−1.9	8.2	0.9	17.2		
26*			−1.0	3.1	−8.0	6.8	−7.9	12.9		

[a] $\phi_1 = c_0(R)+c_1(R)a|a|$; $\phi_2 = \arcsin a$; where a is given by Eqs. (34–37).
[b] Representative trajectories marked (*) were used in designing c_0 and c_1.

acceptable. The remaining trajectories, 8, 17, and 22, are all high target ballistic coefficient and high crossing-angle intercepts which are difficult for any guidance law. Even in those cases it has been found that the errors reduce with range sufficiently fast that acceptable miss-distances are still obtained. Note that Eq. (36) is, with minor exceptions, significantly more accurate than is Eq. (33) on the trajectories with larger errors (trajectories 2, 5, 8, 11, 14, 17, 20, and 22). In order to employ the guidance law structure of Eq. (36) in place of that of Eq. (33) it is necessary to perform the additional work of generating representative trajectories and performing the least-squares fit. This effort is made worthwhile in most cases because the resulting law is more accurate, and it requires less computation in real time.

Real Time Implementation of Multipoint Guidance

Figure 5 shows a functional flow chart detailing the multipoint

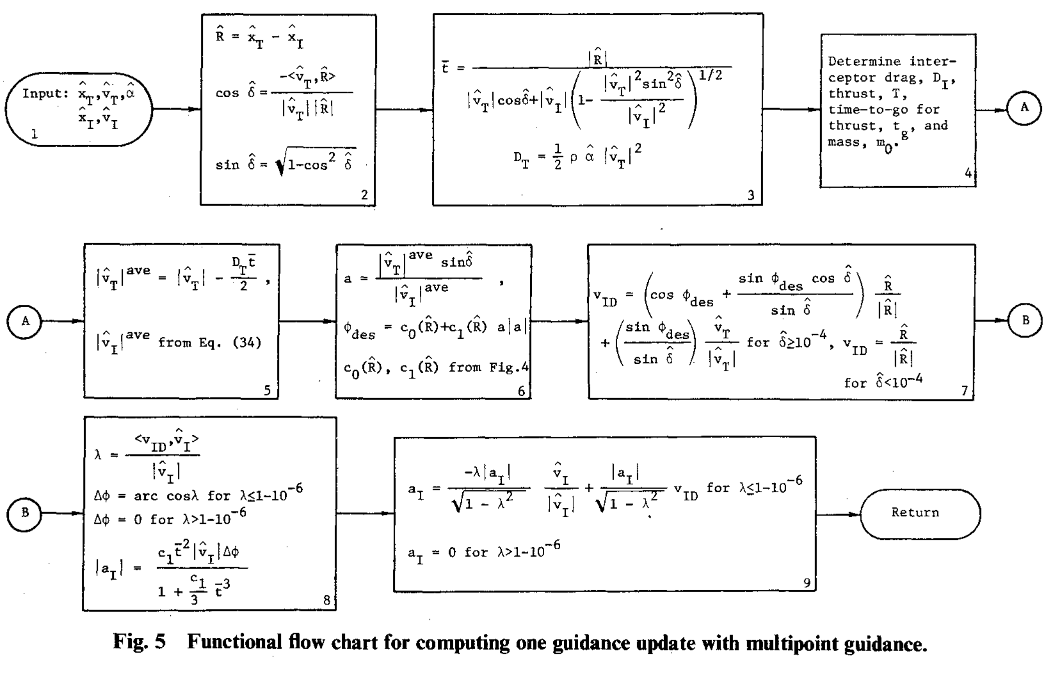

Fig. 5 Functional flow chart for computing one guidance update with multipoint guidance.

Table 3 Breakdown of instructions (equivalent adds) per guidance update for multipoint guidance and a typical predictive guidance law

Function	Instructions/Update	
	Predictive guidance	Multipoint guidance
Coordinate transformation (position and velocity from radar coordinates to ground coordinates)	750	...
Target prediction (table look-up)	1500	...
Interceptor prediction (modified Taylor expansion)	6200	...
Update of intercept time	500	440
Evaluation of predicted miss	400	...
Evaluation of ϕ_{des}	...	450
Evaluation of guidance command	600	20
Guidance component resolution	50	100
Total	10,000	1,010

guidance calculations required for one guidance update. The equations have been developed for an interceptor having lateral control only. In this particular command guidance application where vehicle states are derived from a ground-based radar the calculations are conveniently performed in the ground-based computer. However, the small amount of storage required for the algorithm and the precomputed functions [$c_0(R)$ and $c_1(R)$ of Fig. 4], as well as the modest amount of real time computation, would allow on-board implementation of multipoint guidance for applications in which the required vehicle state estimates are available on-board.

Some explanatory comments follow for the calculations in Fig. 5, which are not directly based on equations derived in the previous sections. The input parameter $\hat{\alpha}$ in box 1 is the latest estimate of the inverse of the target ballistic coefficient. The parameter ρ used in box 3 is the estimated atmospheric air density at the present target altitude. The equation for v_{ID} in box 7 is derived from the facts that v_{ID} is a vector lying in the plane of the vectors $\hat{R}$ and $\hat{v}_T$, it is offset from $\hat{R}$ by the angle ϕ_{des}, and it is a unit vector. The equation for a_I in box 9 is derived from the facts that a_I lies in the plane of v_{ID} and $\hat{v}_I$, has magnitude $|a_I|$, and is a lateral acceleration (i.e., $\langle a_I, \hat{v}_I \rangle = 0$). To avoid numerical difficulties, the equations for v_{ID} and a_I are modified as shown when certain variables are small.

It is obvious that the amount of real time computation required to perform the calculations of Fig. 5 is small in comparison with that of a predictive guidance law which uses accurate numerical integration to predict the target and interceptor trajectories. However, it is not always necessary to perform accurate predictions in order to achieve acceptable miss-distances. Accordingly, a number of approximate predictive guidance laws have been developed which use a variety of approximate methods to predict the trajectories. Table 3 compares the computer instructions required per guidance update for multipoint guidance with those required for a typical approximate implementation of predictive guidance. This particular predictive guidance law uses a table look-up procedure to predict the target trajectory, and a modified Taylor expansion to predict the interceptor trajectory. In spite of these efforts to reduce predictive guidance real time computation, the multipoint guidance law still requires only a tenth of the instructions per update required for predictive guidance.

Conclusions

The design approach that has been employed in developing multipoint guidance has two phases: identification of a good guidance policy without regard to the difficulties of implementation; and design of a real time implementation which approximates the policy. For guidance problems in which the target and interceptor have significant but predictable accelerations the policy of predictive guidance has been widely and effectively employed (for example, predictive guidance is the basis of most of the current laws for intercept of a ballistic re-entry vehicle). The paper presents the theory of predictive guidance and outlines the standard method of its application. Since predictive guidance requires numerical prediction of the target and interceptor trajectories whenever a guidance command is calculated it is costly to implement in terms of real time data processing. This has motivated development of a precomputed form of predictive guidance called multipoint guidance. This guidance law which requires no real time prediction of trajectories is structured during the design phase to approximate the guidance policy defined by predictive guidance at multiple points in the state space. Simulation experiments have shown that multipoint guidance provides miss-distances comparing closely with those of predictive guidance over a wide range of parameters, including time-varying target ballistic coefficients, low and high guidance update rates, noisy radar measurements, and various geometries of intercept. The advantage of multipoint guidance is that real time implementation is simple, typically requiring very little computation.

Possible extensions of the present multipoint guidance law are: 1) incorporating control of axial interceptor accelerations as provided by controlled time of boost ignition or various methods of controlling drag, and 2) extension of multipoint guidance to the interception of maneuvering vehicles. The latter extension first requires determination of an appropriate guidance law policy, ideally obtained from solution of a pursuit-evasion game accurately modeling the goals and capabilities of the target and the interceptor.

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